INVARIANTS VIA WORD FOR CURVES AND FRONTS

NOBORU ITO

Abstract. We construct the infinite sequence of invariants for curves in surfaces by using word theory that V. Turaev introduced. For plane closed curves, we add some extra terms, e.g. the rotation number. From these modified invariants, we get the Arnold's basic invariants and some other invariants. We also express how these invariants classify plane closed curves. In addition, we consider other classes of plane curves: long curves and fronts.

1. Introduction

The object of the paper will be to construct some invariants of plane curves and fronts, and so it is to show one of the method for applying word theory to plane curves and fronts. V. Turaev introduces word theory ([Tu3], [Tu4], [Tu5]). We can consider that this word theory is effective in two view points as follows.

(1) Word is the universal object of knot, curve, etc.
(2) We can treat knots and curves collectively and algebraically, so that we can systematically study in these invariants themselves and relationships among them.

In terms of (1), Turaev applies topological methods (Reidemeister move, homotopy, etc.) to a semigroup consisting of letters, so that creates word which has property as (1) [Tu3]. In terms of (2), Turaev considers equivalent classes of words corresponding to knots or curves, and constructs invariants of knots, for example, Jones polynomial and $\underline{\alpha}$-kei which is similar to kei for knots [Tu4].

For immersed plane closed curves, H. Whitney classified plane closed curves regular homotopically by winding number, which is also called index or rotation number [W]. Long afterword, V. I. Arnold created three basic invariants of plane closed curves J^+, J^-, St by the similar method to knots of V. A. Vassiliev [Va] and classified plane closed curves which have same index ([Ar1], [Ar2]). Arnold also obtained a natural generalization of J^+ to fronts ([Ar1], [Ar3]). Relating to this, M. Polyak systematically reconstructed the Arnold's basic invariants via Gauss diagram and related basic invariants to the Vassiliev invariant [P].

In this paper, by using word theory, we will reconstruct the Arnold's basic invariants and construct some other invariants for plane closed curves, long curves, and fronts. We also express how these invariants classify these plane curves and fronts.

The outline of each section is as follows. In Section 2, we will compose invariants $\{I_n\}$ ('**i**'nvariant of degree '**n**') of curves on a surface. In Section 3, we will construct invariants of plane closed curves CI_n ('**c**'losed curve '**i**'nvariant of degree'**n**') for I_n. CI_2 has the same strength as the Arnold's basic invariants. CI_3 is independent of CI_2. There is an example that two curves take the same values of index, the Arnold's

basic invariants and HOMFLY polynomial of immersed plane closed curves [CGM] but take different values of CI_3. In Section 4, Section 5, we study in long curves and fronts by using the similar technique.

Conventions. In this paper, all surfaces and curves are oriented. For a given surface φ, a *closed curve* (resp. *long curve*) is an immersion : S^1 (resp. $\mathbf{R}$) $\to \varphi$ (resp. $\mathbf{R}^2$) where all of the singular points are transversal double points. A *front* is an immersion : $S^1 \to \mathbf{R}^2$ with the coorientation (defined in Sect. 5.1) where all of the singular points are transversal double points or cusps. (We will precisely define a *front* in Sect. 5.1 .) A *curve* is a closed curve, a long curve, or a front. A *smooth curve* is a closed curve or a long curve. When a curve stands for a closed curve or a front, a *base point* is a point on the curve except on the double points and the cusps. A *pointed curve* is a closed curve or a front endowed with a base point. Winding number (rotation number) is called *index* in this paper.

Acknowledgment– I am grateful to Professor Jun Murakami for giving me numerous fruitful comments. This paper is nothing but trying to answer in my way to his question hitting the mark : "Can we apply word theory to plane curve theory?"

2. Invariants $\{I_n\}$

In this section we equip Turaev's word to construct the sequence of invariants for pointed surface curves.

2.1. **Turaev's word.**

We follow the notation and terminology of [Tu4]. An *alphabet* is a set and its elements are called *letters*. A *word of length* $m \geq 1$ in an alphabet $\mathcal{A}$ is a mapping $\hat{m} = \{1, 2, \ldots, m-1, m\} \to \mathcal{A}$. A word $w : \hat{m} \to \mathcal{A}$ is encoded by the sequence of letters $w(1)w(2)\cdots w(m)$. Two words w, w' are *isomorphic* if there is a bijection $w' = fw$.

A word w is called a *Gauss word* if each element of $\mathcal{A}$ is the image of precisely two elements of $\hat{m}$. For an alphabet α, an α-*alphabet* $\mathcal{A}$ is an alphabet endowed with a projection $|\ | : \mathcal{A} \to \alpha$. An *étale word over* α is a pair $(\mathcal{A}, w)$ where $\mathcal{A}$ is α-alphabet and $w : \hat{m} \to \mathcal{A}$. In this paper, we only treated étale word $(\mathcal{A}, w)$ where w is a surjection. In particular, a *nanoword over* α is an étale word $(\mathcal{A}, w)$ over α where w is a Gauss word. For $(\mathcal{A}, w)$, we admit that we use the simple description 'w' if this w means $(\mathcal{A}, w)$ clearly. An *isomorphism* of α-alphabets $\mathcal{A}_1$, $\mathcal{A}_2$ is a bijection $f : \mathcal{A}_1 \to \mathcal{A}_2$ endowed with $|f(A)| = |A|$ for all $A \in \mathcal{A}_1$. Two nanowords $(\mathcal{A}_1, w_1)$, $(\mathcal{A}_2, w_2)$ over α are *isomorphic* if there is an isomorphism f of α-alphabets $\mathcal{A}_1$, $\mathcal{A}_2$ such that $w_2 = fw_1$.

Until Sect. 4, we denote by *sign* the projection $|\ |$. We also define an alphabet α, an involution τ, and a set S by

$$\alpha = \{-1, 1\}, \quad \tau : -1 \mapsto 1, \quad S = \{(-1,-1,-1), (1,1,1)\}.$$

until Sect. 4.

The following fundamental theorem is established by Turaev [Tu4].

Theorem 2.1. (Turaev) *Every pointed closed curve is represented as a nanoword.*

Proof. For a given pointed closed curve which has precisely m double points, we name the double points $A_1, A_2, \dots, A_m$ along the curve orientation from the base point. Each point precisely corresponds to either -1 or 1 in Figure 1. $\square$

-1 $\qquad\qquad$ 1

FIGURE 1. Two patterns of the double points correspond to two letters in α.

Remark 2.1. Theorem2.1 implies that there is a mapping from a smooth curve Γ on a surface to a nanoword. Conversely, there is a mapping from a nanoword to a smooth curve on a surface. In other words, we determine a unique smooth curve Γ and a unique surface Σ on which Γ is by using the following Theorem2.2.

Theorem 2.2. (Turaev) Let w be a nanoword of length n and $g(\Gamma)$ the minimum genus of the compact surface Σ without boundary such that a pointed smmoth curve Γ is on Σ. There is a mapping $w \mapsto \Gamma$ and

$$g(\Gamma) = 1 + \frac{n - card(\bar{n}/t\theta)}{2}.$$

The construction of the mapping $w \mapsto \Gamma$ is well known and therefore this is omitted (cf. [Tu1], [Tu2], [Vi]). We can calculate $g(\Gamma)$ by the proof of Theorem 9.1.1 in [Tu2] for Turaev's *chart*.

2.2. **Construction of invariants I_n.** For every Gauss word v and every nanoword w, we determine a number $\langle v, w \rangle$. When a nanoword $(\mathcal{A}_w, w)$ over α is given, we consider a sub-word v' of w. If a sub-word v' is Gauss word, we can naturally consider the nanoword $(\mathcal{A}_{v'}, v')$ over α such that $\mathcal{A}_{v'} \subset \mathcal{A}_w$. Therefore for every nanoword $(\mathcal{A}_w, w)$ over α and for every Gauss word v, we can define the mapping by

$$\langle v, w \rangle := \sum_{\text{a sub-word } v' \text{ of } w \text{ isomorphic to } v} \prod\nolimits_{A \in A_{v'}} signA.$$

Let W_n be the free $\mathbf{Q}$-module generated by the set of all of the isomorphism class of the Gauss words where each length of the Gauss word is $2n$. For a given integer d, let N_d be the free $\mathbf{Q}$-module generated by the set of nanowords over α where each length of the nanoword is less than $2d+1$. Expanding $\langle v, w \rangle$ bilinearly, we can make a bilinear mapping $\langle , \rangle$ from $W_n \times N_d$ to $\mathbf{Q}$. For an arbitrary surface, let w_Γ stand for a word which is determined by a curve Γ on the surface.

Theorem 2.3. *The following $\{I_n\}$ (invariant of degree n) is the sequence of surface isotopy invariants for pointed curves on a surface.*

$$I_n(\Gamma) = \left\langle \sum_k x_k v_k, w_\Gamma \right\rangle \qquad (n \in \mathbf{N})$$

where $\{v_k\}$ is the basis of W_n and each x_k is a parameter.

Proof. By using Theorem 2.1, the way of constructing $\langle , \rangle$ implies this theorem. $\square$

2.3. **Generalization of I_n.** We can generalize I_n by introducing *a dimension of a letter.*

Definition 2.1. (a dimension of a letter, a dimension of a word) Let X, $\dot{X}$ be letters. For an arbitrary letter A, we denoted by $d(A) \in \{1, 2\}$ *a dimension of a letter* A which is defined by the following. Let X be a 1-dimensional letter where $d(X) = 1$ and let $\dot{X}$ a 2-dimensionl letter where $d(\dot{X}) = 2$. Next, let $\mathcal{A}$ be an alphabet. For every word $w : \hat{m} \to \mathcal{A}$, the dimension $d(w)$ of word w is defined by $d(w) := \sum_{X \in \mathcal{A}} d(X)$.

The concept of the dimension of word affect on the module 'W_n'. That is why we must redefine 'W_n'.

The following *word space* $\mathcal{W}_n$ is the canonical generalization of 'W_n' defined in Sect. 2.2.

Definition 2.2. (word space) The *word space $\mathcal{W}_n$ of degree n* is the free $\mathbf{Q}$-module generated by the set of all of the n-dimensional Gauss words which may contain 2-dimensional letters.

Replacing W_n defined in Sect. 2.2 with $\mathcal{W}_n$, we can easily check that the similar results are established and can easily generalize Sect. 2.2.

For an arbitrary $(\mathcal{A}_w, w)$ and $v \in \mathcal{W}_n$, we think of $(\mathcal{A}_{v'}, v')$ where v' is a sub-word of w which is isomorphic to v and $\mathcal{A}_{v'} \subset A_w$. We *redefine* $\langle , \rangle$ by

$$\langle v, w \rangle := \sum_{\text{a sub-word } v' \text{ of } w \text{ isomorphic to } v} \prod_{A \in A_v} (signA)^{d(A)}.$$

Corollary 2.1. *The following $\{GI_n\}$ (generalized I_n) is the sequence of surface isotopy invariants for pointed curves on a surface.*

$$GI_n(\Gamma) = \left\langle \sum_k x_k v_k, w_\Gamma \right\rangle \qquad (n \in \mathbf{N})$$

where $\{v_k\}$ is the basis of the word space W_n and each x_k is a parameter.

Remark 2.2. For every plane closed curve Γ, $\left\langle \dot{X}\dot{X}, w_\Gamma \right\rangle$ is the number of the double points for Γ [P].

3. Application of I_n to plane closed curves

We will consider an application of I_n to plane closed curve. We can apply word theory to plane curve theory because word is universal for knots and curves. In this section we will apply it to plane closed curve for example. Plane curves are not only fundamental objects but also proper objects to think of some various applications of word theory. In fact, we can apply word theory to closed curves, long curves, and fronts (Sect. 4, 5). When we apply word theory to plane closed curves, we get some invariants CI_n ('**c**'losed curve '**i**'nvariant degree n). In order to construct CI_n, to add to Turaev's word, we need one more material about plane curve theory : the Arnold's basic invariants defined in the next subsection.

3.1. **The Arnold's basic invariants.** We consider regular homotopy classes of plane curves. Let us rewrite the Arnold's the invariants via Turaev's word theory. To redefine the Arnold's basic invariants [Ar2], we define *elementary moves* that are local moves (Figure 2, 3) of plane curves apart from a base point.

Definition 3.1. (elementary move) Let x, y, z be words that consist of the letter in $\mathcal{A}$ where $\mathcal{A} \cap \{A, B\} = \emptyset$. Elementary move II^+ and elementary move II^- (Figure 2) are defined by

$$II^+ : (\mathcal{A}, xyz) \to (\mathcal{A} \cup \{A, B\}, xAByABz) \text{ if } \tau(signA) = signB,$$
$$II^- : (\mathcal{A}, xyz) \to (\mathcal{A} \cup \{A, B\}, xAByBAz) \text{ if } \tau(signA) = signB.$$

Let x, y, z, t be words that consist of the letter in $\mathcal{A} - \{A, B, C\}$. Elementary move III (Figure 3) is defined by

$$(\mathcal{A}, xAByACzBCt) \to (\mathcal{A}, xBAyCAzCBt) \text{ for } (signA, signB, signC) \in S.$$

The positive elementary moves is the above direction, the negative elementary move is the inverse direction.

Figure 2. Positive elementary move II^+ (left figure) and Positive elementary move II^- (right figure).

For this elementary moves, the Arnold's basic invariants J^+, J^-, St are invariants of curve can be defined by following (cf. [Ar2]).

Definition 3.2. (the Arnold's basic invariants) J^+ is increased by 2 under positive elementary move II^+ but not change under the other, J^- is decrease by 2 under positive elementary move II^- but not change under the other, St is increased by 1 under positive elementary move III but not change under the other, and satisfy the following conditions

$$J^+(K_0) = 0, \quad J^-(K_0) = -1, \quad St(K_0) = 0,$$

FIGURE 3. Elementary move *III*. If the order of three branch are 1, 2, 3 from the left, the direction of the positive move is from the left figure to the right figure. If the order of three branch are 3, 2, 1 from the left, the direction of the positive move is from the right figure to the left figure.

$$J^+(K_{i+1}) = -2i, \quad J^-(K_{i+1}) = -3i, \quad St(K_{i+1}) = i$$

for the base curves $\{K_i\}$ defined by Figure 4.

K_0 K_1 K_2 K_3 K_4

FIGURE 4. Base curve $\{K_i\}$.

3.2. **Construction of invariants $\overline{CI}_n$.** In this subsection, we compose a mapping $[,]$ to construct invariants $\overline{CI}_n$ of plane closed curves. To add to $\langle , \rangle$, we equip *cyclic equivalent* to construct a mapping $[,]$.

Definition 3.3. (cyclic equivalent) Let x be $w(2) \cdots w(m)$ of $w = w(1)w(2) \cdots w(m)$, for two arbitrary Gauss words $w, w' \in W_n$, the relation $\sim$ is defined by

$$w \sim w' \overset{\text{def}}{\Longleftrightarrow} w = Ax \text{ and } w' = -xA.$$

This relation $\sim$ is called *cyclic equivalent.*

The cyclic equivalent is equivalent relation. Let $\overline{W}_n$ be a module consisting of cyclic equivalent classes of the elements of W_n (defined by Sect. 2.2). For $[w] \in \overline{W}_n$, the number of the residue system of $[w]$ is even. That is because $w \sim -w$ implies $w \sim 0$ if this number is odd.

The mapping

$$[,] : \overline{W}_n \times N_d \to \mathbf{Q}$$

is defined by

$$[[v], w] := \langle v_1 + v_2 + \cdots + v_{2l}, w \rangle$$

where $v_1, v_2, \ldots, v_{2l}$ consist of all elements of $[v]$.

Proposition 3.1. *Let Γ be an arbitrary curve. For every $[w] \in \overline{W}_n$, $[[w], w_\Gamma]$ is a surface isotopy invariant of curves.*

Proof. Base point move (Figure 5) is , that is to say, to replace $(\mathcal{A}, w_\Gamma) = AxAy$ with $xA_\tau yA_\tau, signA_\tau = \tau(signA)$ where x, y are consist of the letters of $\mathcal{A} - \{A\}$. Therefore under the base point move, a part of $\langle AxAy, w_\Gamma\rangle$ multiplied -1 is added

FIGURE 5. Base point move.

to $\langle xAyA, w_\Gamma\rangle$. By definition of cyclic equivalence, and by the new numbering $v_1, v_2, \ldots, v_{2l}$ if necessary, we can have

$$v_1 \overset{\text{base point move}}{\longrightarrow} v_2 \overset{\text{base point move}}{\longrightarrow} \cdots \overset{\text{base point move}}{\longrightarrow} v_{2l} \overset{\text{base point move}}{\longrightarrow} v_1.$$

therefore, the value of $\langle v_1 + v_2 + \cdots + v_{2l}, w_\Gamma\rangle$ is not change by base point move. $\square$

Corollary 3.1. *The following $\{\overline{I}_n\}$ is the sequence of surface isotopy invariants for closed curves on a surface.*

$$\overline{I}_n(\Gamma) := \left[\sum_k x_k[v_k], w_\Gamma\right] \qquad (n \in \mathbf{N}).$$

$\{[v_k]\}$ is the base of $\overline{W}_n$, and each x_k is a parameter.

In particular, for every closed plane curve, the following $\{\overline{CI}_n\}$ is the sequence of plane isotopy invariants.

$$\overline{CI}_n(\Gamma) := \overline{I}_n(\Gamma) + f(i) \qquad (n \in \mathbf{N})$$

where f is function of index i.

Next subsection, we introduce CI_2 and CI_3 which are made of $\overline{CI}_2$ and $\overline{CI}_3$.

3.3. CI_2 **and** CI_3**.** For every curve Γ, let i be index and n the number of the double points, we define CI_2 by

$$CI_2(\Gamma; s, t, u) := sn + \langle tXXYY - tXYYX + uXYXY, w_\Gamma\rangle + \frac{t}{2} - \frac{t}{2}i^2.$$

Remark 3.1. $\overline{CI}_2(\Gamma) = [t[XXYY], w_\Gamma]$, and then, we have

$$CI_2(\Gamma; s, t, u) = \overline{CI}_2(\Gamma) + sn + \langle uXYXY, w_\Gamma\rangle + \frac{t}{2} - \frac{t}{2}i^2.$$

M. Polyak proved that $\langle XYXY, w_\Gamma\rangle$ does not depend on the choice of a base point (cf. Theorem 1 proof in [P]). Therefore the invariant CI_2 is well-defined. CI_2 is also not depend on the orientation of the curve Γ because this formula is symmetric. $CI_2(\Gamma; s, t, u)$ is substituted by $CI_2(\Gamma)$ if this means $CI_2(\Gamma; s, t, u)$ clearly. Similarly, for other invariants in this paper we admit the abbreviation like this if its meaning is clear.

Theorem 3.1. (Polyak) *CI_2 is an invariant of plane curves which is as strong as the triple of the three Arnold's basic invariants (J^+, J^-, St). (The definitions of J^+, J^-, St are in* [Ar1], [Ar2], [P]. *)*

Proof. By using Polyak's formulation of the Arnold basic invariants [P], the triple of the three Arnold's basic invariants $(J^+(\Gamma), J^-(\Gamma), St(\Gamma))$ is equal to

$$\left(CI_2(\Gamma; -\frac{1}{2}, 1, -3), CI_2(\Gamma; -\frac{3}{2}, 1, -3), CI_2(\Gamma; \frac{1}{4}, -\frac{1}{2}, \frac{1}{2})\right)$$

and three vectors $(-\frac{1}{2}, 1, -3), (-\frac{3}{2}, 1, -3), (\frac{1}{4}, -\frac{1}{2}, \frac{1}{2})$ are linearly independent. These two facts imply this theorem. □

Remark 3.2. Index is independent of the basic invariants (cf. Figure 6).

FIGURE 6. CI_2 is $3s + \frac{3}{2}t$ on both curves, but the value of index is 0 for the left figure, 2 for the right figure.

We represent

$$XYXYZZ - YXYZZX + XYZZXY - YZZXYX + ZZXYXY - ZXYXYZ,$$

as $[XYXYZZ]$ and represent $XXYYZZ - XYYZZX$ as $[XXYYZZ]$.

For every curve Γ, let i be index, and we define CI_3 by

$$CI_3(\Gamma; s, t) := [s[XYXYZZ] + t[XXYYZZ], w_\Gamma] + i.$$

Remark 3.3. $CI_3(\Gamma; s, t) = \overline{CI}_3(\Gamma; s, t) + i$.

Theorem 3.2. *CI_3 is an invariant of plane closed curves.*

Proof. To prove this, we must prove that $CI_3(\Gamma)$ is not varied by an arbitrary base point move (Figure 5) for every closed curve Γ, but it is immediately concluded by Proposition 3.1. □

There exist two curves such that the values of the HOMFLY polynomial of immersed plane curves [CGM], index, basic invariants are the same; however, the value of CI_3 on one curve is different from that on the other (Figure 7).

This example implies the following.

Corollary 3.2. *CI_3 is independent of index, the Arnold's basic invariants and the HOMFLY polynomial of immersed plane curves.*

As can be seen from the examples above (the case of CI_2 and CI_3), we can get some invariants by the normalization of $\overline{CI}_n$. We denote by CI_n a normalized invariants which is made of $\overline{CI}_n$.

FIGURE 7. Two curves such that the value of the HOMFLY polynomial is $x^2 z_2$ and (i, J^+, J^-, St) is $(2, -4, -7, 2)$. The value of CI_3 is $t + 2$ for the left figure, $-t + 2$ for the right figure.

Corollary 3.3. *Suppose $-\Gamma$ has only difference of orientation from Γ, and let Γ^r be the reflection of Γ,*

$$CI_3(-\Gamma) = -CI_3(\Gamma), \quad CI_3(\Gamma^r) = -CI_3(\Gamma).$$

Remark 3.4.

$$CI_2(-\Gamma) = CI_2(\Gamma).$$

3.4. **Strengthening CI_n.** We can strengthen CI_n by the similar method of GI_n in Sect. 2.3. We must define *marked cyclic equivalent* which is the canonical generalization of cyclic equivalent defined in Sect. 3.2.

Definition 3.4. (marked cyclic equivalent) Let x be $w(2) \cdots w(m)$ of $w = w(1)w(2) \cdots w(m)$, for two arbitrary Gauss words $w, w' \in W_n$, relation $\sim$ is defined by

$$w \sim w' \overset{\text{def}}{\Longleftrightarrow} \begin{cases} w = Ax \text{ and } w' = -xA & \text{if } d(A) = 1 \\ w = \dot{A}x \text{ and } w' = x\dot{A} & \text{if } d(A) = 2 \end{cases}$$

This relation $\sim$ is called *marked cyclic equivalent*.

Replacing cyclic equivalent by *marked cyclic equivalent*, we can easily check that the similar results are established and can easily generalize Sect. 3.2. Therefore we only see the case of CI_3.

For every curve Γ, let i be index, and we define GCI_3 by

$$GCI_3(\Gamma; s, t, u) := \Big[s[XYXYZZ] + t[XXYYZZ] + u[\dot{X}\dot{X}YY], w_\Gamma \Big] + i.$$

Remark 3.5. $GCI_3(\Gamma; s, t, u) = CI_3(\Gamma; s, t) + \Big[u[\dot{X}\dot{X}YY], w_\Gamma \Big]$.

Corollary 3.4. *GCI_3 is an invariant of plane curves.*

Example 3.1. There exist two curves such that the value of CI_3, index, basic invariants are the same; however, the value of GCI_3 on one curve is different from that on the other (Figure 8).

In particular, this example implies the following.

Corollary 3.5. *GCI_3 is a stronger invariant than CI_3.*

Figure 8. Two curves such that (CI_3, i, J^+, J^-, St) is (0, 0, 0, -5, 0). The value of GCI_3 is 0 for the left figure, $-8u$ for the right figure.

4. Application of I_n to long curves

4.1. Construction of invariants LI_n. When we treated long curves via word theory, the following theorem is basic and fundamental.

Theorem 4.1. *Every long curve is represented as a nanoword.*

Proof. Regard $-\infty$ on x-axis as a base point and repeat the proof of Theorem 2.1. $\square$

Let i be index. By Theorem 4.1, we get the sequence of invariants of long curves $\{LI_n\}$ defined by $LI_n = I_n + i$.

Remark 4.1. For an arbitrary function of index i, $I_n + f(i)$ is plane isotopy invariant.

4.2. The basic invariants of long curves. In similar way of defining the Arnold's basic invariants of plane closed curve in Sect. 3.1, we define the basic invariants of long curves in this subsection (cf. [GN], [ZZP]).

Definition 4.1. (basic invariants of long curves) J^+ is increased by 2 under positive elementary move II^+ but not change under the other, J^- is decrease by 2 under positive elementary move II^- but not change under the other, St is increased by 1 under positive elementary move III but not change under the other, and satisfy the following conditions

$$J^+(L_i) = -|i|, \quad J^-(L_i) = -2|i|, \quad St(L_i) = \frac{1}{2}|i|$$

for the base curves $\{L_i\}$ defined by Figure 9.

L_{-3} L_{-2} L_{-1} L_0 L_1 L_2 L_3

Figure 9. Base curve $\{L_i\}$.

4.3. LI_2 **and** LI_3**.** For long curve L, let i be index and n the number of double points, we define LI_2 by

$$LI_2(L; s, t, u, v) := sn + \langle tXXYY + uXYYX + vXYXY, w_L\rangle - \frac{t}{2}i^2$$

As case of CI_2, LI_2 is not depend on the orientation of long curve L.

Theorem 4.2. *LI_2 is an invariant of plane curves which is as strong as (J^+, J^-, St). In other words, for two arbitrary long curves L_1, L_2,*

$$LI_2(L_1) = LI_2(L_2) \Longleftrightarrow (J^+(L_1), J^-(L_1), St(L_1)) = (J^+(L_2), J^-(L_2), St(L_2)).$$

Before we begin proving Theorem4.2, we will prove Lemma 4.1. (Similar formula is concluded in case closed curves [P].)

Lemma 4.1.

$$LI_2(L; \frac{1}{2}, 1, 1, 1) = \frac{n}{2} + \langle XXYY + XYYX + XYXY, w_L\rangle - \frac{i^2}{2} \equiv 0$$

In particular, left side is independent of elementary move II and III.

Proof. For $w_L = (\mathcal{A}, w_L)$, we have $i = \sum_{A\in\mathcal{A}} signA$. Therefore

$$i^2 = \left(\sum_{A\in\mathcal{A}} signA\right)^2 = \left\langle \dot{X}\dot{X} + 2XXYY + 2XYYX + 2XYXY, w_L \right\rangle.$$

The number of double points n is equal to $\left\langle \dot{X}\dot{X}, w_L \right\rangle$. □

Next, we will prove Theorem 4.2. It is sufficient that we prove the following.

Proof. ($\Longrightarrow$). The following three relations are concluded by Proposition 4.1.

Proposition 4.1.

$$\begin{aligned} J^+(L) &= LI_2(L; -\frac{1}{2}, 1, -1, -3), \\ J^-(L) &= LI_2(L; -\frac{3}{2}, 1, -1, -3), \\ St(L) &= LI_2(L; \frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}). \end{aligned}$$

(Proof of Proposition 4.1.) Let n be the number of double points and $f(i)$ a function on index i. By using Theorem 2.3,
$I_2(L; s, t, u, v) = ns + \langle tXXYY + uXYYX + vXYXY\rangle - t\frac{i^2}{2}$
is an invariant of long curves.
By definition of J^+, J^-, St,

$$\begin{aligned} J^+(L) &= LI_2(L; s, 2s + 2, 2s, 2s - 2), \\ J^-(L) &= LI_2(L; s, 2s + 4, 2s + 2, 2s), \\ St(L) &= LI_2(L; s, 2s - 1, 2s, 2s). \end{aligned}$$

These $J^+(L), J^-(L), St(L)$ satisfies

$$J^+(L_i) = -|i|\,, \quad J^-(L_i) = -2\,|i|\,, \quad St(L_i) = \frac{1}{2}\,|i|\,.$$

Especially, In the case $s = -\frac{1}{2}$ on J^+, in the case $s = -\frac{3}{2}$ on J^-, in the case $s = -\frac{1}{4}$ on St, these are still the basic invariants. We have thus proved the Proposition 4.1 that implies ($\Longleftarrow$).

($\Longleftarrow$). For two arbitrary long curves L_1, L_2, assume

$$(J^+(L_1), J^-(L_1), St(L_1)) = (J^+(L_2), J^-(L_2), St(L_2)).$$

Four vectors $(\frac{1}{2}, 1, 1, 1)$, $(-\frac{1}{2}, 1, -1, -3)$, $(-\frac{3}{2}, 1, -1, -3)$, $(\frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ are linearly independent. By adding Lemma 4.1 and Proposition 4.1, the assumption implies $LI_2(L_1) = LI_2(L_2)$. $\square$

Remark 4.2. Index is independent of the basic invariants. (cf. Figure 10).

FIGURE 10. LI_2 is $4s + t - 3u$ on both curves, but the value of index is 0 for the left figure, 2 for the right figure.

Remark 4.3. If n of them are not same, the values of LI_2 for two curves are not the same because $J^+ - J^- = n$.

For every long curve L, let i be index, we define LI_3 by

$$\begin{aligned} LI_3(L; x_1, x_2, \ldots, x_{14}, x_{15}) \;:=\; & \langle x_1 XYXYZZ + x_2 XYXZZY + x_3 XYZZXY \\ & + x_4 XYYZXZ + x_5 XXYZYZ + x_6 XYZYZX \\ & + x_7 XYYXZZ + x_8 XXYZZY + x_9 XYZZYX \\ & + x_{10} XYZYXZ + x_{11} XYXZYZ + x_{12} XYZXZY \\ & + x_{13} XYZXYZ + x_{14} XXYYZZ + x_{15} XYYZZX, \\ & w_L\rangle + i. \end{aligned}$$

Theorem 4.3. *LI_3 is an invariant of long curves.*

Proof. Theorem 2.3 immediately deduces this conclusion. $\square$

Corollary 4.1. *Let L^r be the reflection of a long curve L, $LI_3(L^r) = -LI_3(L)$.*

Remark 4.4. $LI_2(L^r) = LI_2(L)$.

The following are examples of several pairs of long curves such that the values of index and three basic invariants are the same, but the value of LI_3 on one long curve is different from that on the other.

L_1 $\qquad$ L_1^r $\qquad$ L_2 $\qquad$ L_2^r

FIGURE 11. Four long curves such that $(i, J^+, J^-, St) = (0, 0, -4, 0)$. $LI_3(L_1) = 2x_7 - x_{14} - x_{15}$, $LI_3(L_1^r) = -2x_7 + x_{14} + x_{15}$, $LI_3(L_2) = 2x_8 - x_{14} - x_{15}$, $LI_3(L_2^r) = -2x_8 + x_{14} + x_{15}$.

Example 4.1. For L_1, L_1^r, L_2, L_2^r (Figure 11) such that (i, J^+, J^-, St) is $(0, 0, -4, 0)$, each value of LI_3 is different from another of them.

Example 4.2. For L_3, L_4, L_5, L_6 (Figure 12) such that (i, J^+, J^-, St) = (2, -4, -8, 2), each value of LI_3 is different from another of them.

L_3 $\qquad$ L_4 $\qquad$ L_5 $\qquad$ L_6

FIGURE 12. Four long curves such that (i, J^+, J^- St)= (2, -4, -8, 2). $LI_3(L_3) = -x_7 - x_9 + 2$, $LI_3(L_4) = -2x_8 + 2$, $LI_3(L_5) = 2x_9 + 2$, $LI_3(L_6) = 2x_{14} + 2$.

4.4. **Strengthening LI_n.** We can strengthen LI_n in the way constructing GI_n in Sect. 2.3.

For every long curve L, let i be index, we define GLI_3 by

$$\begin{aligned} GLI_3(L; x_1, x_2, \dots, x_{20}, x_{21}) \; := \; & LI_3(L; x_1, x_2, \dots, x_{14}, x_{15}) \\ & + \langle x_{16}\dot{X}\dot{X}YY + x_{17}\dot{X}YY\dot{X} + x_{18}XX\dot{Y}\dot{Y} \\ & + x_{19}X\dot{Y}\dot{Y}X + x_{20}X\dot{Y}X\dot{Y} + x_{21}\dot{X}Y\dot{X}Y, w_L\rangle. \end{aligned}$$

Corollary 4.2. *GLI_3 is an invariant of long curves.*

Example 4.3. There exist two curves such that the value of LI_3, index, basic invariants are the same; however, the value of GLI_3 on one curve is different from that on the other (Figure 13).

FIGURE 13. Two curves such that (LI_3, i, J^+, J^-, St) is (0, 0, 0, -4, 0). The value of GLI_3 is 0 for the left figure, $-2x_{17} + 2x_{19}$ for the right figure.

In particular, this example implies the following.

Corollary 4.3. *GLI_3 is a stronger invariant than LI_3.*

4.5. **The Arnold-type invariant of degree 3.**

Definition 4.2. (The Arnold-type invariant of degree 3) Let J^+-type invariant be an invariant do not change under elementary move II^-, III, J^--type invariant be an invariant do not change under elementary move II^+, III, St-type invariant be an invariant do not change under elementary move II^+, II^-.

J^+-type invariant, St-type invariant are available from GLI_3. These are expressed J_3^+ (J^+ of degree 3), St_3 (St of degree 3).

Theorem 4.4. *Let L be long curve.*

$$\begin{aligned} J_3^+(L) &= GLI_3(L; s+t-v, s-t+u, -s+t+v, -s+3t-u, s+t-v, \\ &\quad 2t-v, s, s, t, u, 2t-v, -2s+4t-u, -2s+2t+v, s-t+v, \\ &\quad \frac{1}{2}t, \frac{1}{2}s, v, \frac{1}{2}s, \frac{1}{2}t, -s+2t-\frac{1}{2}u, \frac{1}{2}u), \\ St_3(L) &= GLI_3(L; 2u, p, 2s, q, 2x, 2y, 2u, 2x, 2y, 2z, r, 2z, 2z, s, t, u, v, x, y, z, z). \end{aligned}$$

Proof. To prove this theorem, we check variations of each parameter's coefficient of GLI_3:

$$\begin{aligned} GLI_3(L; x_1, x_2, \ldots, x_{20}, x_{21}) &:= LI_3(L; x_1, x_2, \ldots, x_{14}, x_{15}) \\ &\quad + \langle x_{16}\dot{X}\dot{X}YY + x_{17}\dot{X}YY\dot{X} + x_{18}XX\dot{Y}\dot{Y} \\ &\quad + x_{19}X\dot{Y}\dot{Y}X + x_{20}X\dot{Y}X\dot{Y} + x_{21}\dot{X}Y\dot{X}Y, w_L\rangle. \end{aligned}$$

Consider variations of each parameter's coefficient under each elementary move. □

5. Application of I_n to fronts

5.1. **The basics of fronts.** To begin with, we recall basic concepts or results about fronts.

Definition 5.1. (contact element) A *contact element* in the plane is a line in the tangent plane (Figure 14).

Figure 14. Contact element.

Definition 5.2. (coorientation) The *coorientation* of a contact element is the choice of the half-plane into which the contact element divides the tangent plane (Figure 14).

The manifold M consisting of all of the contact elements in the plane is diffeomorphic to solid torus. Consider immersion : $S^1 \to M$. For each element of this curve in the plane, the coorient of the element is determined (Figure 14). This curve which is Legendrian submanifold of M is called Legendrian curve, the image of projection to plane of this curve is called a *front*.

By using the following several concepts: elementary move (Definition 5.7), index (Definition 5.5), and Maslov index (Definition 5.6), a front is regarded as a plane curve generated by $K_{i,k}$ (Figure 15) via elementary moves because Theorem 5.1 is established.

$2k$ cusps $\qquad$ $i-1$ kinks $\quad$ $2k$ cusps

$K_{0,k}$ $\qquad$ $K_{i,k}$

FIGURE 15. Base fronts $\{K_{i,k}\}$.

Let the alphabet α_*, the involutions τ_1, and τ_2, the set S_* be

$$\begin{aligned}
\alpha_* &= \{a_+, a_-, b_+, b_-\}, \\
\tau_1 &: a_+ \mapsto b_+, a_- \mapsto b_-, \\
\tau_2 &: a_+ \mapsto b_-, a_- \mapsto b_+, \\
S_* &= \{(a_\epsilon, a_\eta, a_\xi), (b_\epsilon, b_\eta, b_\xi) : \epsilon, \eta, \xi = +, \text{ or } -\}.
\end{aligned}$$

(They are different from the involution τ, S_* for α_* in [Tu4].)

To the following relations

$$a_+ + b_+ = 0, \quad a_- + b_- = 0$$

are established, we consider the ring $\overline{\mathbf{Q}\tilde{\alpha_*}}$ into which two-sided ideal generated by $a_+ + b_+$, $a_- + b_-$ divides the monoid algebra $\mathbf{Q}\tilde{\alpha_*}$ where $\tilde{\alpha_*}$ is commutative monoid generated by $\alpha_* \cup \{1\}$.

Remark 5.1. If we regard a_+, b_+, τ_1 in the same light as a, b, τ the discussion from Sect. 2 to Sect. 4 still holds on this establishment because a_-, b_-, τ_2 do not appear Sect. 2, 3, and 4. That is to say, this establishment is canonical generalization of the establishment in Sect. 2, 3, and 4.

From this section, we denote by $|\ |$ projection : $\mathcal{A} \to \alpha_*$.

Definition 5.3. ($|A|$ corresponding to a double point) We define $|A|$ corresponding to a double point A by Figure 16.

a_+ a_- b_+ b_-

FIGURE 16. Correspondence 4 patterns of double points to 4 alphabets.

a_+ a_- b_+ b_-

FIGURE 17. Correspondence 4 patterns of cusps to 4 alphabets.

Definition 5.4. ($|K|$ corresponding to a cusp) $|K|$ where K is a cusp is defined by Figure 17.

Definition 5.5. (index) *Index* of a front F is the number of the full rotations of the coorienting normal vector counter clockwise while we trip along the front once.

Definition 5.6. (Maslov index) For étale word $(\mathcal{A}_F, w_F)$ of front F, let K is a letter of $\mathcal{A}_F$ for a cusp. Maslov index μ is defined by

$$\mu = card\{K \in \mathcal{A}_F ||A| = a_+ \text{ or } b_+\} - card\{K \in \mathcal{A}_F ||A| = a_- \text{ or } b_-\}.$$

The local moves of fronts (Figure 18, Figure 19, Figure 20, Figure 21, Figure 22) is defined by the following. Suppose the local moves are admitted apart from base point.

Definition 5.7. (elementary move of fronts) Let x, y, z be words that consist of the letter in $\mathcal{A}$. Four kinds of elementary move II are defined by

$$\begin{aligned}
SII^+ &: (\mathcal{A} \cup \{A, B\}, xAByABz) \to (\mathcal{A}, xyz) \text{ with } \tau_1(|A|) = |B|,\\
SII^- &: (\mathcal{A}, xyz) \to (\mathcal{A} \cup \{A, B\}, xAByBAz) \text{ with } \tau_1(|A|) = |B|,\\
DII^+ &: (\mathcal{A}, xyz) \to (\mathcal{A} \cup \{A, B\}, xAByABz) \text{ with } \tau_1(|A|) = |B|,\\
DII^- &: (\mathcal{A} \cup \{A, B\}, xAByBAz) \to (\mathcal{A}, xyz) \text{ with } \tau_1(|A|) = |B|.
\end{aligned}$$

Let x, y, z, t be words that consist of the letter in $\mathcal{A} - \{A, B, C\}$.
elementary move III is defined by

$$III : (\mathcal{A}, xAByACzBCt) \to (\mathcal{A}, xBAyCAzCBt) \text{ for } (|A|, |B|, |C|) \in S_*.$$

Let x, y, z be words that consist of the letter in $\mathcal{A} - \{K\}$.

Two kinds of elementary move Π is defined by

$$\Pi^{+} : (\mathcal{A}, xKyz) \to (\mathcal{A} \cup \{A, B\}, xAKByABz)$$
$$\text{or } (\mathcal{A}, xKyz) \to (\mathcal{A} \cup \{A, B\}, xAByAKBz) \text{ with } \tau_2(|A|) = |B|,$$
$$\Pi^{-} : (\mathcal{A} \cup \{A, B\}, xAKByBAz) \to (\mathcal{A}, xKyz)$$
$$\text{or } (\mathcal{A} \cup \{A, B\}, xAByBKAz) \to (\mathcal{A}, xKyz) \text{ with } \tau_2(|A|) = |B|.$$

Let x, y be words that consist of the letter in $\mathcal{A}$. elementary move Λ is defined by

$$\Lambda : (\mathcal{A}, xy) \to (\mathcal{A} \cup \{A, K_1, K_2\}, xAK_1K_2Ay) \text{ with } \tau_1 \circ \tau_2(|K_1|) = |K_2|.$$

The positive elementary moves is the above direction, negative is the inverse direction.

FIGURE 18. Elementary move *SII*$^{+}$ (left figure), *SII*$^{-}$ (right figure).

FIGURE 19. Elementary move *DII*$^{+}$ (left figure), *DII*$^{-}$ (right figure).

FIGURE 20. Elementary move Π^{+}.

Remark 5.2. *SII* stands for safe 2-move,and *DII* means dangerous 2-move (2-move, 3-move resemble Reidemeister move *II*, *III*). In the lift of the plane, self-tangency under dangrous move is corresponded to crossing of the Legendrian curve (cf. [Ai2]).

We have the next theorem due to Gromov [Gr].

FIGURE 21. Elementary move Π^-.

FIGURE 22. Elementary move Λ.

Theorem 5.1. (Gromov) *Any front whose index is i, Maslov index is $2k$ is deformed via DII^+, DII^-, SII^+, SII^-, III, Π^+, Π^-, Λ from $K_{i,k}$ (Figure 15).*

The object of this section is giving a classification of fronts more detail than the classification by Theorem 5.1. To do this, we consider an application a method like Sect. 3, 4 to front which may have not only double points but also cusps. We consider only double points and cusps due to Theorem 5.1.

We get following theorem.

Theorem 5.2. *All fronts are represented as étale words.*

Proof. For an arbitrary given front which has precisely n double points and precisely m cusps, we name double points $A_1, A_2, \ldots, A_n$ and name cusps $K_1, K_2, \ldots, K_m$ along front from the base point. Every double point and every cusp precisely corresponds to a unique element of α_* in Figure 16. □

5.2. **The basic invariants of fronts.** The basic invariants of fronts J^+, J^-, St are three invariants can be defined as follows (cf. [Ar2], [P]).

Definition 5.8. J^+ is increased by 2 under positive elementary move DII^+, DII^- but not change under the other,

J^- is decreased by 2 under positive elementary move SII^+, SII^- but not change under the other,

St is increased by 1 under positive elementary move III, increased by $\frac{1}{2}$ under positive elementary move Π^+, and decreased by $\frac{1}{2}$ under positive elementary move Π^-, but not change under the other, and satisfy the following conditions

$$J^+(K_{0,k}) = -k, \quad J^-(K_{0,k}) = -1, \quad St(K_{0,k}) = \frac{k}{2},$$

$$J^+(K_{i+1,k}) = -2i - k, \quad J^-(K_{i+1,k}) = -3i, \quad St(K_{i+1,k}) = i + \frac{k}{2}$$

for the base curves $\{K_i\}$ defined by Figure 15.

5.3. **Construction of invariants FI_n.** In this subsection we will construct invariants of fronts.

Definition 5.9. (fake Gauss word) We call a word w a *fake Gauss word of dimension n* if $w' : 2m \to \mathcal{A}$ is a Gauss word where w' is a sub-word of w and $\{w(1), w(2), \ldots, w(2m)\} \sqcup \{K_1, K_2, \ldots, K_{2l}\} = \{w(1), w(2), \ldots, w(2(m+l))\}$ where $n = m + 2l$.

Definition 5.10. (fake nanoword) An étale word $(\mathcal{A}, w)$ over α_* is a *fake nanoword over α_* of dimension n* if $(\mathcal{A}, w)$ satisfies $(\mathcal{A} - \{K_1, K_2, \ldots, K_{2l}\}, w')$ is an nanoword over α_* where the length of w is $card\mathcal{A}' + 2l = n$ and w' is a sub-word of w.

Remark 5.3. We can consider a front F on a surface which is a curve on a surface with coorientation and cusps. A fake nanoword gives rise to a nanoword w by neglecting letters for cusps. We can calculate the genus $g(F)$ of a surface on which a fronts because $g(F)$ is equal to $g(\Gamma)$ where Γ is determined by the nanoword w by using Theorem 2.2.

Definition 5.11. (fake word space) The fake word space of degree n is the word space generated by fake Gauss words of dimension n.

In 5.3 and 5.4, suppose fake Gauss word, fake nanoword and fake word space are made of only 1-dimensional letters. We denote by FW_n the fake word space is made of fake Gauss words such that all letters are 1-dimensional letters.

Let ϵ be $\epsilon = +$, or $-$. For every $A \in \alpha_*$-alphabet $\mathcal{A}$, $sign(|A|)$ is 1 if $|A| = b_\epsilon$ and $sign(|A|)$ is -1 if $|A| = a_\epsilon$ (We consider its generalization in Sect. 6). Let w_F stand for a word which is determined by a front F. By using Theorem 2.3, for every pointed front (which means every front with a base point), the following $\{FI_n\}$ is plane isotopy invariants sequence.

$$FI_n(F) = \left\langle \sum_k x_k v_k, w_F \right\rangle \qquad (n \in \mathbf{N}).$$

v_k is a sequence consisting of all elements in FW_n, and each x_k is a parameter.

In the same way of Sect. 3.2, in order to simplify following description, we define proper equivalent classes of fake nanowords over α_*.

Definition 5.12. (cyclic equivalent for fronts) Let $w(2) \cdots w(n)$ of $w = w(1)w(2) \cdots w(n)$ represent x, for two arbitrary $w, w' \in FW_n$, relation $\sim$ is defined by

$$w \sim w' \overset{def}{\Longleftrightarrow} \begin{cases} w = Ax \text{ and } w' = -xA & \text{if A means a double point} \\ w = Kx \text{ and } w' = xK & \text{if when K means a cusp} \end{cases}$$

This relation $\sim$ is called the *cyclic equivalent for fronts.*

Corollary 5.1. *The cyclic equivalent for fronts is equivalent relation.*

Let $\overline{FW}_n$ be a module consisting of cyclic equivalent classes for front and FN_d the $\mathbf{Q}$-module generated by the set of fake nanoword over α_* $\{(\mathcal{A}, w)\}$ such that $card\mathcal{A}$ is less than $d + 1$.

The mapping $[,] : \overline{FW}_n \times FN_d \to \mathbf{Q}$ is defined by

$$[[v], w] := \langle v_1 + v_2 + \cdots + v_{2l}, w_F \rangle ,$$

$v_1, v_2, \ldots, v_{2l}$consist of all elements of $[v]$.

Proposition 5.1. *For every $[w]$ which is one of base of $\overline{FW}_n$, $[[w], w_F]$ is an invariant of curve Γ.*

Proof. The proof is similar to the proof of Proposition 3.1. □

5.4. FI_2 **and** FI_3**.** For every front $F = (\mathcal{A}_F, w_F)$, let i be index and $2c$ the number of cusps,

$$n_\epsilon = card\{A \in \mathcal{A}_F : |A| = a_\epsilon \text{ or } b_\epsilon\} \ (\epsilon = +, \text{or } -).$$

We define FI_2 by

$$\begin{aligned} FI_2(F; p, q, r, s, t, u, v) \ := \ & pn_+ + qn_- + \langle rXXYY - rXYYX + sXYXY \\ & + tKXX + tXXK - tXKX + uKK, w_F \rangle + vc \\ & + \frac{r}{2} - \frac{r}{2} i^2. \end{aligned}$$

FI_2 is also not depend on the orientation of each front F because this formula is symmetric.

Theorem 5.3. *FI_2 is an invariant which is stronger than(J^+, J^-, St).*
i.e. For a front F, $FI_2(F)$ is not depend on the choice of the base point, for two arbitrary fronts F_1, F_2,

$$FI_2(F_1) = FI_2(F_2) \Longrightarrow (J^+(F_1), J^-(F_1), St(F_1)) = (J^+(F_2), J^-(F_2), St(F_2)),$$

and the converse can not be established.

Proof. (I) First, we will prove that for an arbitrary front F, $FI_2(F)$ is not depend on the choice of a base point. Base point move (Figure 24) means x, y is word consisting of letters in $\mathcal{A} - \{A\}$, in the case of A is a double point, as Proposition 3.1, $(\mathcal{A}, w_F) = AxAy \to xA_\tau yA_\tau, |A_\tau| = \tau_1(|A|)$, in the case of A is a cusp $i.e. A = K$, $(\mathcal{A}, w_F) = Kxy \to xyK, |K|$ is not change .

$\langle XXYY - XYYX, w_F \rangle , \langle XYXY, w_F \rangle$ is not depend on the choice of a base point due to [P].

Therefore to prove it is checking increase and decrease of only the terms

$$\langle tKXX + tXXK - tXKX, w_F \rangle$$

under an arbitrary base point move.

(II) For front F_1, F_2,

$$FI_2(F_1) = FI_2(F_2) \Longrightarrow (J^+(F_1), J^-(F_1), St(F_1)) = (J^+(F_2), J^-(F_2), St(F_2)),$$

and the converse can not be established.

($\Longrightarrow$). The following three relations are concluded by Polyak in [P].

$$\begin{aligned}
J^+(F) &= FI_2(F; -\frac{1}{2}, -\frac{3}{2}, 1, -3, \frac{1}{2}, \frac{1}{4}, -\frac{3}{4}), \\
J^-(F) &= FI_2(F; -\frac{3}{2}, -\frac{1}{2}, 1, -3, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}), \\
St(F) &= FI_2(F; \frac{1}{4}, \frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \frac{3}{8}).
\end{aligned}$$

($\Longleftarrow$ can not be established.) There exists counterexample : Figure 23.

FIGURE 23. The values of basic invariants of two fronts are same, but the value of FI_2 of one of them is different from that of another.

$\square$

FIGURE 24. Base point move.

Remark 5.4. There is a relation $J^+ - J^- = n_+ - n_- - c$ (cf. [P]).

We will consider application I_3 to fronts.

Let i be index. For every front F, we define FI_3 by

$$FI_3(F; x, y, z, p, q, r, s, t) := CI_3(F; x, y, z) + \Big[p[XKXYY] + q[KXXYY] + r[XKYXY] + s[XXKK] + t[KKK], w_F\Big].$$

Theorem 5.4. *FI_3 is an invariant of fronts.*

Proof. To prove this, we must prove that $FI_3(F)$ is independent of base point move (Figure 24) for every front, but it is immediately concluded by Proposition 5.1. $\square$

Corollary 5.2. *Suppose $-F$ has only difference of orientation from F, and let F^r be the reflection of F,*

$$FI_3(-F) = -FI_3(F), \quad FI_3(F^r) = -FI_3(F).$$

Remark 5.5. $FI_2(-F) = FI_2(F)$.

The following is example of the pair of curves such that the values of index, Maslov index, the basic invariants, and FI_2 are the same, but the value of FI_3 on one front is different from that on the other.

Example 5.1. There exist two curves (Figure 25), such that (i, μ, J^+, J^-, St, FI_2) = (1, 2, -3, -4, 6, $2p - r - u + v$) and the values of one front is different from another.

FIGURE 25. Two fronts such that (i, μ, J^+, J^-, St, FI_2) = (1, 2, -3, -4, 6, $2p - r - u + v$). The value of FI_3 is $2s + 1$ for left figure, $2s - 4p + 1$ for right figure.

5.5. **Strengthening FI_n.** In this section, suppose fake Gauss word (Definition 5.9), fake nanoword (Definition 5.10), and fake word space (Definition 5.11) are made of not only 1-dimensional letters but also 2-dimensional letters. We can strengthen FI_n by the similar method of Sect. 2.3, 3.4, 4.4. We must define *marked cyclic equivalent for fronts* which is the canonical generalization of cyclic equivalent for fronts defined in Sect. 5.3.

In distinction from FW_n, we denote by $\mathcal{FW}_n$ the fake word space may have fake Gauss words which letters are not only 1-dimensional letters but also 2-dimensional letters.

Definition 5.13. (marked cyclic equivalent for fronts) Let $w(2)\cdots w(n)$ of $w = w(1)w(2)\cdots w(n)$ represent y, for two arbitrary $w, w' \in \mathcal{FW}_n$, relation $\sim$ is defined by

$$w \sim w' \overset{def}{\Longleftrightarrow} \left\{ \begin{array}{lll} \text{when X means a double point,} & w = Xy \text{ and } w' = -yX & \text{if } d(X) = 1 \\ & w = \dot{X}y \text{ and } w' = y\dot{X} & \text{if } d(X) = 2 \\ \text{when K means a cusp,} & w = Ky \text{ and } w' = yK & \text{if } d(K) = 1 \\ & w = \dot{K}y \text{ and } w' = y\dot{K} & \text{if } d(K) = 2 \end{array} \right.$$

This relation $\sim$ is called *marked cyclic equivalent for fronts.*

Replacing cyclic equivalent for fronts by *marked cyclic equivalent for fronts*, we can easily check that the similar results are established and can easily generalize Sect. 5.4. Therefore we only see the case of FI_3. Let i be index. For every front F, we define GFI_3 by

$$GFI_3(F; x, y, z, p, q, r, s, t, u, v, h) := GCI_3(F; x, y, z) + \Big[p[XKKXYY] + q[KXXXYY]$$
$$+ r[XKYXY] + s[KKK] + t[XXKK] + u[\dot{K}XX] + v[K\dot{X}\dot{X}] + h[\dot{K}K], w_F\Big].$$

Corollary 5.3. *GFI_3 is an invariant of fronts.*

Example 5.2. There exist two fronts such that the value of FI_3, index, Maslov index, basic invariants are the same; however, the value of GFI_3 on one curve is different from that on the other (Figure 26).

FIGURE 26. Two fronts such that (FI_3, i, μ, J^+, J^-, St) is ($2q+2$, 2, 0, -4, -5, 2). The value of GFI_3 is $2q-2u+2v+2$ for the left figure, $2q+2u+2$ for the right figure.

In particular, this example implies the following.

Corollary 5.4. *GFI_3 is a stronger invariant than FI_3.*

6. Generalization

By replacing the function *sign* with ρ, we get more general invariants.

6.1. **Generalization of I_n.** Let $\tilde{\alpha}$ be commutative monoid generated by an alphabet α which contains the unit element and consider monoid algebra $\mathbf{Q}\tilde{\alpha}$. For every $A \in \alpha$-alphabet $\mathcal{A}$, suppose mapping $\rho : |A| \to \rho(|A|) \in \mathbf{Q}\tilde{\alpha}$ is given.

When an arbitrary étale word w is given, choose a sub-word v of w, for this sub-word v, sub-étaleword $(\mathcal{A}_v, v)$ of w is determined , and then

$$\rho(v) := \prod_{A \in A_v} \{\rho(|A|)\}^{d(A)}$$

can be defined. By using this, for every étale word w and every sub-word v of w, we define the mapping

$$\langle v, w \rangle := \sum_{\text{a sub-word } v' \text{ of } w \text{ isomorphic to } v} \rho(v').$$

Let W_n be the $\mathbf{Q}$-module generated by the set of all of the fake-Gauss words in FW_n. Let FN_d be the free $\mathbf{Q}$-module generated by the set of fake nanowords over α $\{(\mathcal{A}, w)\}$ such that $card\mathcal{A}$ is less than $d+1$. Expanding $\langle v, w \rangle$ bilinearly, we can make a bilinear mapping$\langle , \rangle$ from $FW_n \times FN_d$ to $\mathbf{Q}$. For an arbitrary surface, let w_Γ stand for a word which is determined by a curve Γ on the surface.

Theorem 6.1. *The following $\{\tilde{I}_n\}$ (invariant of degree n) is the sequence of surface isotopy invariants for pointed curves on a surface.*

$$\tilde{I}_n(\Gamma) = \left\langle \sum_k x_k v_k, w_\Gamma \right\rangle \qquad (n \in \mathbf{N})$$

where $\{v_k\}$ is the basis of W_n and each x_k is a parameter.

Proof. Theorem 2.3 and the construction of $\langle , \rangle$ deduce immediately this theorem. $\square$

6.2. **Investigation of the generalization in the case FI_2.** For example, we will consider the case FI_2, and so we get an invariant $\widetilde{FI_2}$ which is stronger than FI_2. For every element $A \in \alpha_*$-alphabet $\mathcal{A}$, ρ is defined by $\rho(|A|) = |A| \in \overline{\mathbf{Q}\widetilde{\alpha_*}}$.

Let F be front. By using the construction of $[,]$, the following $\widetilde{FI_n}$ is a plane isotopy invariant of fronts.

$$\widetilde{FI_n}(F) = \left[\sum_{i,j} x_j [v_i], w_F \right] \qquad (n \in \mathbf{N}).$$

$[v_i]$ in $\overline{FW}_n$ (cf. Proposition 5.1), each x_j is a parameter.

For front $F = (\mathcal{A}_F, w_F)$, let i be index, $2c$ the number of cusps,

$$n_\epsilon = card\{A \in \mathcal{A}_F : |A| = a_\epsilon \text{ or } b_\epsilon\} \ (\epsilon = +, \text{ or } -).$$

We define $\widetilde{FI_2}$ by

$$\begin{aligned}\widetilde{FI_2}(F; p, q, x, z, t, v, r) \ &:= \ pn_+ + qn_- + \langle xXXYY - xXYYX + zXYXY \\ &\quad +tKXX + tXXK - tXKX + vKK, w_F\rangle + rc \\ &\quad +\frac{x}{2} - \frac{x}{2} i^2.\end{aligned}$$

Theorem 6.2. *$\widetilde{FI_2}$ is an invariant which is stronger than FI_2.*

Proof. The value of FI_2 is obtain from $\widetilde{FI_2}$ by regarding a_+, a_- as 1. Therefore $\widetilde{FI_2}$ is at least as strong as FI_2. So, Figure 27 deduces that $\widetilde{FI_2}$ is an invariant which is stronger than FI_2.

FIGURE 27. The values of FI_2 for these two fronts are p $+q$ $+x$ $+2t$ $-v$ $+2r$. The values of $\widetilde{FI_2}$ for left figure is $p + q + (-\frac{3}{2} + a_+ a_-)x + (a_-{}^2 + a_+{}^2)t + (a_+ a_-)v + 2r$, The values of FI_2 for right figure is $p + q + (-\frac{3}{2} + a_+ a_-)x + (a_+{}^2 + 2a_+ a_- - a_-{}^2)t + (a_+ a_-)v + 2r$.

$\square$

Remark 6.1. Suppose let F^r be the reflection of F, and then $\widetilde{FI_2}(F) = \widetilde{FI_2}(F^r)$. On the other hand, suppose $-F$ has only the difference of orientation from F, and then there exists an example (left figure in Figure 27) as $\widetilde{FI_2}(F) \neq \widetilde{FI_2}(-F)$ (cf. CI_2, LI_2).

Remark 6.2. For these two fronts, $(i, \mu, J^+, J^-, St) = (2, 0, -2, -1, 2)$, and in terms of invariants of fronts : $f^+, f^-, p^\uparrow, p^\downarrow, \lambda^\uparrow, \lambda^\downarrow$ due to Aicardi [Ai1], (f^+, f^-, $p^\uparrow$, $p^\downarrow$, $\lambda^\uparrow$, $\lambda^\downarrow$) = (2, 0, −2, 0, 2, 0).

Remark 6.3. $\widetilde{FI_2}$ is the deformation $\overline{FI_2}$. Moreover, because $\left\langle \dot{X}\dot{X}, w_F \right\rangle = n_+{a_+}^2 + n_-{a_-}^2$, if the term $\left\langle \dot{X}\dot{X}, w_F \right\rangle$ is taken place of $n_+p + n_-q$, the strength of the invariant does not change.

References

[Ar1] V. I. Arnold, *Plane curves, their invariants, perestroikas and classifications*, Advances in Soviet Mathematics, AMS **21**, Providence (1994), 33–91.

[Ar2] V. I. Arnold, *Topological invariants of plane curves and caustics*, University Lecture Series **5**, Providence, RI, 1994.

[Ar3] V. I. Arnold, *Invariants and perestroikas of plane fronts*, Proc. Steklov Inst. Math. **209** (1995), 11–56.

[Ai1] F. Aicardi, *Discriminants and local invariants of planar fronts*, The Arnold-Gelfand mathematical Seminars, Birkhäuser Boston, Boston, MA (1997), 1–76.

[Ai2] F. Aicardi, *Invariant polynomial of framed knots in the solid torus and its application to wave fronts and Legendrian knots*, Journal of Knot Theory and Its Ramifications, **5** (1996), 743–778.

[CGM] S. Chumutov, V. Goryunov, H. Murakami, *Regular Legendrian knots and the HOMFLY polynomial of immersed plane curves*, Math. Ann. **317** (2000), 389–413.

[Gr] M. Gromov, *Partial differential relations*, Springer-Verlag. Berlin and New York, 1986.

[GN] S. M. Gusein-Zade, S. M. Natanzon, *The Arf-invariant and the Arnold invariants of plane curves*, The Arnold-Gelfand mathematical Seminars, Birkhäuser Boston, Boston, MA (1997), 267–280.

[P] M. Polyak, *Invariants of curves and fronts via Gauss diagrams*, Topology **37** (1998), 989–1009.

[Tc] V. Tchernov, *Arnold-type invariants of wave fronts on surfaces*, Topology **41** (2002), 1–45.

[Tu1] V. Turaev, *Curves on surface, charts, and words*, Geometriae Dedicata **116** (2005), 203–236.

[Tu2] V. Turaev, *Virtual strings*, Ann. Inst. Fouruer 54 **7** (2004), 2455-2525.

[Tu3] V. Turaev, *Topology of words*, math.CO/0503683.

[Tu4] V. Turaev, *Knots and words*, math.GT/0506390.

[Tu5] V. Turaev, *Lectures on topology of words*, RIMS-1567 (2006).

[Va] V. A. Vassilliev, *Cohomology of knot spaces*, Theory of singularities and its applications, Adv. Soviet Math. **1**, Amer. Math. Soc. , Providence, RI (1990), 23–69.

[Vi] O. Viro, *Virtual links, Orientations of chord diagrams and Khovanov homology*, Proceedings of 12^{th} Gk̈ova Geometry-Topology Conference (2005), 184–209.

[W] H. Whitney, *On regular closed curves in the plane*, Compositio Math. **4** (1937), 276–284.

[ZZP] J. Zhou, J. Zou, J. Pan, *The basic invariants of long curve and closed curve perestroikas*, Journal of Knot Theory and Its Ramifications, **7** (1998), 527–548.

Department of Mathematics and Applied Mathematical Science Sciences
Graduate School of Fundamental Science and Engineering
Waseda University

3-4-1 Okubo, Shinjyuku-ku
Tokyo 169-8555, JAPAN
email: noboru@moegi.waseda.jp